Harmonic Functions

Definition. A real valued function $u(x,y)$ of two variables defined on a region is harmonic if it has continuous partial derivatives of second order and satisfies the Laplace equation $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$.

ie $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

We shall denote $\frac{\partial u}{\partial x}$ by u_x, $\frac{\partial u}{\partial y}$ by u_y, $\frac{\partial^2 u}{\partial x^2}$ by u_{xx} and $\frac{\partial^2 u}{\partial y^2}$ by u_{yy}.

Example 1: Let $u(x,y) = x+y$ Then
$u_x = 1 = u_y$ and $u_{xx} = 0 = u_{yy}$.
Clearly u_x, u_y, u_{xx}, u_{yy}, u_{xy} are continuous
and $u_{xx} + u_{yy} = 0$
Thus u is harmonic.

Example 2: Let $u(x,y) = xy$. Then $u_x = y$, $u_y = x$,
$u_{xx} = 0 = u_{yy}$, $u_{xy} = 1 = u_{yx}$.

Clearly u satisfies the conditions required for it to be a harmonic function.

Now we shall consider a non-trivial example.

Example 3: Let $u(x,y) = \log (x^2+y^2)^{1/2}$.

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$$u_x = \frac{1}{(x^2+y^2)^{1/2}} \cdot \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

Similarly $u_y = \frac{y}{x^2+y^2}$

Now $u_{xx} = \frac{1}{x^2+y^2} - \frac{x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

Similarly $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$$u_{xy} = x \cdot \left(\frac{-1}{(x^2+y^2)^2} \cdot 2y\right) = \frac{-2xy}{(x^2+y^2)^2} = u_{yx}$$

These partial derivatives are continuous for $(x,y) \neq (0,0)$

Further, $u_{xx} + u_{yy} = 0$

Thus u is harmonic on $\mathbb{C} - \{0\}$.

Fact: If $f(z) = u + iv$ is analytic, then u and v are harmonic.

We have derivatives of analytic function are analytic. Hence u and v have continuous partial derivatives of second order (recall that $f'(z) = u_x + iv_x$ hence $f''(z) = u_{xx} + iv_{xx}$. Using Cauchy Riemann equations one can see that other partial derivatives are continuous). Also $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$

Hence $u_{xx} + u_{yy} = 0$ $(v_{yx} = v_{xy})$

and u is harmonic. Similarly, v is harmonic

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Exercise: Identify $u(x,y)$ in examples (1) and (2) as real/imaginary part of an analytic function.

$u(x,y) = x+y = Re\, f(z)$, where $f(z) = (1-i)z$

which is analytic.

$u(x,y) = xy = Re\left(-i\frac{z^2}{2}\right)$

Exercise: Show that $\sin x \cosh y$ and $\cos x \sinh y$ are harmonic functions.

Definition (Conjugate of a harmonic function)

Let $u(x,y)$ be harmonic. Then a harmonic function $v(x,y)$ is said to be a conjugate of u if $u_x = v_y$ and $u_y = -v_x$.

Remarks: (1) If v is a harmonic conjugate then $v+c$, where c is a constant, is also a harmonic conjugate of u.

(2) Harmonic function need not have a conjugate

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a continuous function $f(z)$ on $\mathbb{C}-\{0\}$ such that $\exp(f(z)) = z$. Suppose such a function exists. Then $f(e^{i\theta}) = i\theta + i2\pi k(\theta)$, where $k(\theta)$ is an integer depending on θ. Since f is continuous it follows that $k(\theta) =$ constant k. But this implies that $f(e^{i0}) = i2\pi k \neq f(e^{i2\pi}) = (k+1)i2\pi$, a contradiction $(e^{i0} = e^{i2\pi})$.

Going back to example (3) we have

$$u(x,y) = \log(x^2+y^2)^{1/2} = \log|z|$$

is harmonic on $\mathbb{C}-\{0\}$. If this has a conjugate then $u(x,y) + iv(x,y) = \log|z| + iv$ is analytic on $\mathbb{C}-\{0\}$. It follows that this a branch of log. By the above discussion this is not possible. Thus u doesn't have a conjugate.

Next, we shall see how to determine a conjugate (if it exists) of a harmonic function.

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Finding conjugate of a harmonic function:

Going back to example (1) and (2), we determine their conjugates

(1) Let $u(x,y) = x+y$. We have observed that $x+y = Re\,(1-i)z$. and Hence it has a harmonic conjugate, namely $im\,(1-i)z = y-x$.

Suppose $V(x,y)$ is a harmonic conjugate of $u(x,y)$

Then $u_x = V_y$ and $u_y = -V_x$.

Now $V = \int V_y\,dy = \int u_x\,dy = \int dy = y + \phi(x)$

where $\phi(x)$ is the constant of integration (integration is with respect to y, hence constant of integration is a function of x).

Now $V_x = -u_y = -1$

i.e, $\phi'(x) = -1$ and hence $\phi(x) = -x + c$, where c is a constant.

Therefore, $V = y - x + c$

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(2) $u(x,y) = xy$. (This is the real part of an analytic function, hence it has a conjugate).

$u_x = y \qquad u_y = x.$

If v is a conjugate of u, then

$v_y = u_x \Rightarrow v = \int u_x dy = \int y\,dy = \frac{y^2}{2} + \phi(x).$

As $v_x = -u_y = -x$, $\phi'(x) = -x$. Hence $\phi(x) = -\frac{x^2}{2} + c$

$\therefore \quad v = \frac{y^2}{2} - \frac{x^2}{2} + c$, c being a constant.

Recall that $u = \mathrm{Re}\left(i\frac{z^2}{2}\right)$

$\mathrm{im}\left(-i\frac{z^2}{2}\right) = \frac{y^2 - x^2}{2}$

$z^2 = x^2 + y^2 + 2ixy$

The method to find conjugate:

Suppose u is harmonic and v is a conjugate.

Then $u_x = v_y$ and $u_y = -v_x$.

Then $v = \int u_x dy + \phi(x)$

Hence $v_x = \frac{d}{dx}\left(\int u_x dy\right) + \phi'(x) = -u_y$

Use this equation to find $\phi(x)$.

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What are the regions G for which every harmonic function has a conjugate.

The answer is: G is a simply connected region

(in example (3) $u(x,y) = \log\sqrt{x^2+y^2}$ is harmonic on $\mathbb{C}-\{0\}$ and $\mathbb{C}-\{0\}$ is not simply connected ($\mathbb{C}_\infty - (\mathbb{C}-\{0\})$ $= \{0\} \cup \{\infty\}$ which is not connected)).

Let G be a simply connected region and u is harmonic on G. Let $F(x,y) = u_x - i u_y$

Then $(u_x)_x = (-u_y)_y$ $(\because u_{xx} + u_{yy} = 0)$.

$(u_x)_y = -(-u_y)_x$ $(\because u_{xy} = u_{yx})$

Thus the real and imaginary parts of $F(x,y)$ satisfy Cauchy Riemann equations. Further, the first order partial derivatives of u_x and $-u_y$ are $u_{xx}, u_{xy}, -u_{yx}, -u_{yy}$ which are continuous as u is harmonic. Hence $F(x,y)$ is analytic.

Let $\gamma \subset G$ be a closed curve. Then by the general form of Cauchy Theorem

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$$\int_\gamma F(x,y)\,dz = 0$$

i.e., $$\int_\gamma (u_x - iu_y)(dx + i\,dy) = 0$$

$$\Rightarrow \int_\gamma u_x\,dx + u_y\,dy + i\int_\gamma u_x\,dy - u_y\,dx = 0$$

$$\Rightarrow \int_\gamma u_x\,dy - u_y\,dx = 0$$

$\therefore$ $\exists$ a function $V(x,y)$ $\ni$

$V_x = -u_y$ and $V_y = u_x$.

Now $V_{xx} = -u_{yx}$ and $V_{yy} = u_{xy}$.

$\therefore$ $V_{xx} + V_{yy} = 0$

Clearly, second order partial derivatives of $V(x,y)$ are continuous.

Hence V is a harmonic conjugate of u.

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Properties of Harmonic Functions

Proposition 1: If $u : G \to \mathbb{R}$ is harmonic, then u is infinitely differentiable.

Proof: Fix $z_0 = x_0 + iy_0$ in G and $r > 0$ be such that $B(z_0, r) \subset G$. As disks are simply connected there exists a function v such that $u + iv$ is analytic on $B(z_0, r)$. It is known that analytic functions are infinitely differential. Hence their real and imaginary parts are infinitely differentiable. Therefore, u is infinitely differentiable on $B(z_0, r)$.

Proposition 2: Let $u : G \to \mathbb{R}$ be a harmonic function and $\bar{B}(a, r)$ (the closed ball with centre at a and radius r) be contained in G. Then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

Proof: Let D be a disk such that $\bar{B}(a, r) \subset D \subset G$ and let f be an analytic function on D with

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Ref - 2.

By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_{\text{circular}} \frac{f(z)}{z-a}\,dz$$

converting to polar coordinates $z = a + re^{i\theta}$

and $dz = ire^{i\theta}\,d\theta$. Therefore, the RHS of the

above equation becomes $\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$

$$\therefore f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

Taking the real parts we get

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta$$

(Real part of integral is the integral of the real part)

Definition 3: A continuous function $u: G \to \mathbb{R}$ has the mean value property (MVP) if whenever $\bar{B}(a,r) \subset G$

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta$$

By proposition 2, every harmonic function satisfies MVP. Converse is also true ie, any function with

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MVP is harmonic. This will be proved in due course.

Proposition A: (Maximum Principle)

Let G be a region and suppose that u is a real valued continuous function on G with MVP. If there is a point $a \in G$ such that $u(a) \geq u(z)$ for all z in G, then u is a constant function.

Proof: Let $A = \{ b \in G \mid u(b) = u(a) \}$

The idea is to show that A is both open and closed. As $a \in A$ ($\Rightarrow A \neq \emptyset$) and G is connected it follows that $A = G$ and this proves u is constant.

A is closed: Let (b_n) be a sequence in A such that $b_n \to b$. Then $u(b_n) \to u(b)$ (u is continuous)

But $u(b_n) = u(a)$ for all n. Hence $u(b) = u(a)$ and $b \in A$. This proves A is closed.

A is open: Let $b \in A$ be arbitrary. Then $B(b, r) \subset G$

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for some $r > 0$.

Claim: $B(b,r) \subset A$.

Suppose the claim is not true. Then there exists $x \in B(b,r)$ such that $x \notin A$. Then $u(x) < u(a)$ ($u(a) \geq u(z)$ $\forall z \in G$). Let $\rho = |b-x|$. Then $x = b + \rho e^{i\theta}$, $0 \leq \theta < 2\pi$. As u is continuous and $u(x) < u(a)$, there exists an interval $I \subset [0, 2\pi]$ containing θ such that $u(b+\rho e^{i\beta}) < u(a)$ for $\beta \in I$.

Hence by MVP

$$u(b) = \frac{1}{2\pi} \int_0^{2\pi} u(b + re^{i\theta})\, d\theta$$

$$< u(a)$$

$$\left[\int_0^{2\pi} u(b+re^{i\theta})\, d\theta = \int_{\theta \in I} u(b+re^{i\theta})\, d\theta + \int_{\theta \notin I} u(b+re^{i\theta})\, d\theta \right.$$

$$< u(a)\, l(I) + u(a)\, l([0,2\pi] - I)$$

$$\left. = u(a) \right]$$

a contradiction. Hence $B(b,r) \subset A$ and A is open.

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Corollary 5 (Minimum Principle): If $u : G \to \mathbb{R}$ is continuous and satisfies MVP and for some $a \in G$ $u(a) \leq u(z)$ for all $z \in G$, then u is constant.

Proof: Note that $-u$ also satisfies MVP.

As $(-u)(z) \leq (-u)(a)$ $\forall z \in G$, by the maximum principle $-u$ is constant. Hence u is constant.

Corollary 6: Let G be a bounded region and suppose that $w : \bar{G} \to \mathbb{R}$ is a continuous function which satisfies MVP on G. If $w(z) = 0$ for all $z \in \partial G$, then $w = 0$ on G.

Proof: As $\bar{G}$ is compact and w is continuous it attains its minimum and maximum on $\bar{G}$. By maximum (minimum) principle these are attained on ∂G. Hence minimum and maximum of w are zero. Hence $w = 0$ on G.

Continued.

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Harmonic Functions on a disk

To start with harmonic functions on open unit disk will be studied. The Poisson kernel plays a vital role in this study.

Defn: (Poisson kernel)

For $0 \le r < 1$ and $-\infty < \theta < \infty$ define

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

Different expressions of $P_r(\theta)$:

(1) Let $z = re^{i\theta}$ where $0 \le r < 1$. Then

$$\frac{1+z}{1-z} = (1+z)(1+z+z^2+\cdots) \quad (\because |z|<1)$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n$$

$$\therefore \operatorname{Re}\frac{1+z}{1-z} = 1 + 2\sum r^n \cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty} r^n \left(e^{in\theta} + e^{-in\theta}\right)$$

$$= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} r^{|n|} e^{in\theta}$$

$$= P_r(\theta).$$

(2) Also

(a)

(b)

(c)

(d)

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2. Also $\dfrac{1+re^{i\theta}}{1-re^{i\theta}} = \dfrac{(1+re^{i\theta})(1-re^{-i\theta})}{|1-re^{i\theta}|^2}$

$$= \frac{1+r(e^{i\theta}-e^{-i\theta})-r^2}{|1-re^{i\theta}|^2}$$

$$= \frac{1-r^2+2ir\sin\theta}{1-2r\cos\theta+r^2} \qquad (1-re^{i\theta})(1-re^{-i\theta})$$

$$\therefore \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

Hence by (1), $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta+r^2}$

Proposition 7: (Properties of $P_r(\theta)$) The Poisson kernel satisfies the following properties:

(a) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$

(b) $P_r(\theta) > 0$ for all θ, $P_r(-\theta) = P_r(\theta)$ and P_r is periodic in θ with period 2π.

(c) $P_r(\theta) < P_r(\delta)$ if $0 \le \delta < |\theta| \le \pi$.

(d) for each $\delta > 0$, $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \ge |\theta| \ge \delta$.

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Proof: (a) Fix r, $0 \le r < 1$, then the series defining $P_r(\theta)$ converges uniformly by Weierstrass M-test

$$\left[\sum_{n=1}^{\infty} \left| r^n e^{in\theta} \right| << \sum_{n=1}^{\infty} r^n \qquad r<1 \right.$$

$$\left. \sum_{n=-\infty}^{-1} \left| r^{|n|} e^{in\theta} \right| << \sum_{|n|=1}^{\infty} r^{|n|} \qquad r<1 \right]$$

$$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} r^{|n|} e^{in\theta} \right) d\theta$$

$$= \frac{1}{2\pi} \cdot 2\pi + \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} r^{|n|} \int_{-\pi}^{\pi} e^{in\theta}\, d\theta$$

$$= 1 \qquad \left(\int_{-\pi}^{\pi} e^{in\theta}\, d\theta = \frac{e^{in\theta}}{in} \Big|_{-\pi}^{\pi} = \frac{1}{in} e^{in\pi} - \frac{1}{in} e^{-in\pi} \right)$$

$e^{in\pi}$ / $e^{-in\pi}$

(b) By expression (2) of $P_r(\theta)$ it follows that $P_r(\theta) > 0$ (since $r<1$). From the same expression it also follows that $P_r(\theta) = P_r(-\theta)$ ($\cos\theta = \cos(-\theta)$) since $\cos\theta$ is periodic in θ of period 2π, the same expression implies that $P_r(\theta)$ is also

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periodic in θ of period 2π.

(c) Let $0 < \delta < \theta \leq \pi$ (by (b) $P_r(\theta) = P_r(-\theta)$)

To prove (c) we have to show $P_r(\beta)$ is decreasing in $[\delta, \theta]$. So consider $P_r(\beta)' = \frac{(1-r^2)(-2r\sin\beta)}{(1-2r\cos\beta+r^2)^2} < 0$

(Since $\sin\beta \geq 0$ for $|\beta| \leq \pi$).

This proves $P_r(\theta) < P_r(\delta)$.

(d) To prove this we have to show that

$$\lim_{r \to 1^-} \left(\sup \{ P_r(\theta) / \delta \leq |\theta| \leq \pi \} \right) = 0.$$

But by (c) $P_r(\theta) \leq P_r(\delta)$. So enough to show

$\lim_{r \to 1^-} P_r(\delta) = 0$ which is clear from expression (2) of $P_r(\theta)$

Dirichlet Problem for unit disk:

The Dirichlet problem is about ~~Laplace~~ solutions of Laplace equation with ~~bc~~ given boundary conditions.

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i.e. if G is a region and $f: \partial G \to \mathbb{R}$ is continuous, then to find a function $u: \bar{G} \to \mathbb{R}$ which is continuous, harmonic in G and satisfies $u(z) = f(z)$ for $z \in \partial G$.

The following theorem states that the Dirichlet problem can be solved for unit disk.

Theorem 8: Let $D = \{z : |z| < 1\}$ and suppose that $f: \partial D \to \mathbb{R}$ is a continuous function. Then there is a continuous function $u: \bar{D} \to \mathbb{R}$ such that

(a) $u(z) = f(z)$ for $z \in \partial D$

(b) u is harmonic in D

Moreover u is unique and is defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) \, dt \qquad (*)$$

for $0 \le r < 1$ and $0 \le \theta \le 2\pi$.

Proof: Define u by the formula given in $(*)$ if $0 < r < 1$ and $u(z) = f(z)$ for $z \in \partial D$

Then $u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) \, dt$ for $0 \le r < 1$

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$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}} f(e^{it})\,dt$$

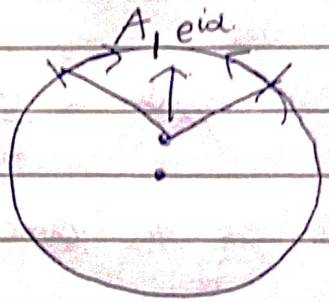
$$= \mathrm{Re}\, \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}} \right) f(e^{it})\,dt$$

It is known that $\int_{-\pi}^{\pi} \left(\frac{e^{it}+z}{e^{it}-z} \right) f(e^{it})\,dt$

is analytic

$\therefore$ u is harmonic in D.

Remains to show u is continuous on $\bar{D}$. As u is harmonic in D if is continuous in D. So we have only to show u is continuous on ∂D.



Take an arbitrary point $e^{i\alpha}$ on ∂D. There are two ways a point in $\bar{D}$ can approach $e^{i\alpha}$ one along the boundary ∂D and the other from D.

Therefore, the continuity of u at $e^{i\alpha}$ requires for $\epsilon > 0$ an r, $0 < r < 1$ and an arc A on ∂D

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containing $e^{i\alpha}$ such that

$|u(\rho e^{i\theta}) - f(e^{i\alpha})| < \epsilon$ for $r < \rho < 1$ and $e^{i\theta} \in A$

To simplify the calculations we assume $\alpha = 0$

(proof in the general case follows by rotation of axes through an angle α)

Since f is continuous at $z = 1$ we get $\delta > 0$ such that

$|f(e^{i\theta}) - f(1)| < \frac{\epsilon}{3}$ whenever $|\theta| < \delta$.

Let $M = \max\{|f(e^{i\theta})| : |\theta| \le \pi\}$.

By (d) of Prop 7 there is r, $0 < r < 1$ such that

$$P_\rho(\theta) < \frac{\epsilon}{3M} \quad \text{for } r < \rho < 1 \text{ and } |\theta| \ge \frac{\delta}{2}$$

Let A be the arc $\{e^{i\theta} : |\theta| < \frac{\delta}{2}\}$

Then for $e^{i\theta} \in A$ and $r < \rho < 1$

$$u(\rho e^{i\theta}) - f(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_\rho(\theta - t)\, dt - f(1)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(e^{it}) - f(1)) P_\rho(\theta - t)\, dt.$$

(use (a) of Prop 7).

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$$= \frac{1}{2\pi}\left[\int_{|t|<\delta} P_\rho(\theta-t)\,(f(e^{it})-f(1))\,dt + \int_{|t|\geq\delta} P_\rho(\theta-t)(f(e^{it})-f(1))\,dt\right]$$

$|\theta - t| \geq ||\theta| - |t|| = |t| - |\theta|$ for $|\theta| < \delta/2$, $|t| \geq \delta$

$$\geq \frac{\delta}{2}$$

$$\therefore |u(\rho e^{i\theta}) - f(1)| \leq \frac{1}{2\pi}\int_{|t|<\delta} P_\rho(\theta-t)\,|(f(e^{it})-f(1))|\,dt$$

$$+ \frac{1}{2\pi}\int_{|t|\geq\delta} P_\rho(\theta-t)\,|f(e^{it})-f(1)|\,dt.$$

$$< \frac{\epsilon}{3} + \frac{2M\cdot\epsilon}{3M} = \epsilon.$$

$$\left[\frac{1}{2\pi}\int_{|t|<\delta} P_\rho(\theta-t)\,|f(e^{it})-f(1)|\,dt < \frac{\epsilon}{3}\,\frac{1}{2\pi}\int_{|t|<\delta} P_\rho(\theta-t)\,dt\right.$$

$$< \epsilon/3 \quad \left(|f(e^{it})-f(1)| < \frac{\epsilon}{3}\right)$$

$$\frac{1}{2\pi}\int_{|t|\geq\delta} P_\rho(\theta-t)\,|f(e^{it})-f(1)|\,dt \leq \frac{1}{2\pi}\int_{|t|\geq\delta} P_\rho(\theta-t)\,(|f(e^{it})|+|f(1)|)\,dt$$

$$\leq 2M\,\frac{1}{2\pi}\int_{|t|\geq\delta} P_\rho(\theta-t)\,dt$$

$$< 2M\,\frac{\epsilon}{3M}\,\frac{1}{2\pi}\int_{|t|\geq\delta} dt < 2M\cdot\frac{\epsilon}{3M}$$

$\left.\left(\text{Since } |\theta-t| \geq \frac{\delta}{2}\right)\right]$.

Uniqueness of u: Suppose v is another such function

Then $u(z) = v(z)$ on ∂D i.e., $(u-v)(z) = 0$ on ∂D

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and $u-v$ is harmonic in D and hence satisfies MVP. Therefore, by corollary 6 $u-v=0$ on $\bar{D}$

$\therefore u=v$ and this proves uniqueness.

Corollary 9: If $u: \bar{D} \to \mathbb{R}$ is a continuous function that is harmonic in D then

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, u(e^{it})\, dt \quad \text{for } 0 \le r < 1$$

and all θ. Moreover, u is the real part of the analytic function $f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})\, dt$.

Proof: By theorem 8 $\exists\ w: \bar{D} \to \mathbb{R}$ such that

$w = u$ on ∂D and w is harmonic in D

By uniqueness we get $w = u$ on $\bar{D}$. As

$$w(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, u(e^{it})\, dt$$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, u(e^{it})\, dt$$

The RHS is the real part of $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})\, dt$

Corollary 10: Let $a \in \mathbb{C}$, $\rho > 0$ and suppose h is a continuous real valued function on $\{z : |z-a| = \rho\}$. Then there is a unique continuous function

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$W : \bar{B}(a,\rho) \to \mathbb{R}$ such that W is harmonic in $B(a,\rho)$ and $W(z) = u(z)$ for all z with $|z-a| = \rho$.

Proof: Define f on ∂D by $f(e^{i\theta}) = h(a+\rho e^{i\theta})$

Then f is continuous on ∂D. Then by theorem 8

$\exists$ a function $u : \bar{D} \to \mathbb{R}$ such that u is continuous on $\bar{D}$ and harmonic in D and $u(e^{i\theta}) = f(e^{i\theta})$

Define W by $W(z) = u\left(\frac{z-a}{\rho}\right)$ on $\bar{B}(a,\rho)$

$W(z)$ is continuous on $\bar{B}(a,\rho)$

Then $W(z) = h(z)$ for $|z-a| = \rho$ and W is harmonic in $B(a,\rho)$.

Theorem 11: If $u : G \to \mathbb{R}$ is a continuous function which has the MVP then it is harmonic

Proof: Let $a \in G$ be arbitrary and $\rho > 0$ be such that $\bar{B}(a,\rho) \subset G$. It is sufficient to show that u is harmonic on $B(a,\rho)$. By corollary 10 there exists a function $W : \bar{B}(a,\rho) \to \mathbb{R}$ such that $W(z) = u(z)$ on $\{z : |z-a| = \rho\}$ and W is harmonic in $B(a,\rho)$

Since W has MVP and $W(z) = u(z)$ for $|z-a| = \rho$

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$w(z) = u(z)$ for $z \in B(a,R)$ by Corollary 6 (applied to $w-u$)

Hence u is harmonic in $B(a,R)$

Poisson kernel for arbitrary disk $B(0,R)$:

This can be obtained by replacing r by $\frac{r}{R}$ in the formula for Poisson kernel for unit disk

Thus $P_r(\theta) = \dfrac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}$ (use expression (2))

for $0 \le r < R$ and all θ.

If u is continuous on $\bar{B}(a,R)$ and harmonic in $B(a,R)$, we get

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, u(a + Re^{it})\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2}\, u(a + Re^{it})\, dt. \quad (4)$$

Note that $|Re^{it} - re^{i\theta}|^2 = (Re^{it} - re^{i\theta})(Re^{-it} - re^{-i\theta})$

$$= R^2 - rR\,e^{i(\theta - t)} - rR\,e^{-i(\theta - t)} + r^2$$

$$= R^2 - 2rR\cos(\theta - t) + r^2$$

Also $R-r$

$\therefore$

i.e.,

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Also $R-r \leq |Re^{it} - re^{i\theta}| \leq R+r$

$$\therefore \frac{R^2-r^2}{(R+r)^2} \leq \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \leq \frac{R^2-r^2}{(R-r)^2}$$

i.e., $$\frac{R-r}{R+r} \leq \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \leq \frac{R+r}{R-r} \quad (*)$$

Harnack's Inequality:

If $u : B(a,R) \to \mathbb{R}$ is continuous, harmonic in $B(a,R)$ and $u \geq 0$, then for $0 \leq r < R$ and all θ

$$\frac{R-r}{R+r} u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r} u(a)$$

Proof: Multiply $*$ throughout by $u(a+Re^{it})$.

As $u \geq 0$ we get

$$\frac{R-r}{R+r} u(a+Re^{it}) \leq \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} u(a+Re^{it}) \leq \frac{R+r}{R-r} u(a+Re^{it})$$

Integrating between $-\pi$ and π and dividing by 2π we get

$$\frac{R-r}{R+r} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a+Re^{it})\,dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} u(a+Re^{it})\,dt$$

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$$\leq \frac{R+r}{R-r}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+re^{it})\,dt$$

i.e. $\frac{R-r}{R+r}\,u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}\,u(a)$

(use: Prop 2 and ↑).

Notation: For an open subset G of $\mathbb{C}$, let $\mathrm{Har}\,G$ denote the set of harmonic functions on G.

Remark. $\mathrm{Har}(G) \subset C(G,\mathbb{R})$ and it inherits the metric on $C(G,\mathbb{R})$.

(Harnack's Theorem): Let G be a region. Then

(a) $\mathrm{Har}(G)$ is complete

(b) If (u_n) is a sequence in $\mathrm{Har}\,G$ such that $u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or (u_n) converges to a harmonic function u in $\mathrm{Har}(G)$.

Proof. (a) Let ~~$B(a;r) \subset$~~

(a) Since $Har\,G \subset C(G,\mathbb{R})$ and $C(G,\mathbb{R})$ is complete enough to show $Har\,G$ is closed in $C(G,\mathbb{R})$. So let (u_n) be a sequence in $Har\,G$ such that $u_n \to u$. We have to show $u \in Har\,G$. For this it is enough to show u satisfies MVP (note that u is continuous).

Let $\bar{B}(a,R) \subset G$. Since u_n is harmonic it satisfies MVP

Therefore,

$$u_n(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(a+Re^{it})\,dt$$

As $u_n \to u$ uniformly on $\bar{B}(a,R)$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(a+Re^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})\,dt.$$

$$\therefore \quad u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})\,dt.$$

Thus u satisfies MVP

(b) We can assume $u_n \geq 0$ (replacing u_n by $u_n - u_1$, if necessary)

Let $u(z) = \sup\{u_n(z),\ n \geq 1\}$

Then either $u(z) = \infty$ or $u(z) < \infty$, in this case $u_n(z) \to u(z)$

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Let $A = \{z / u(z) = \infty\}$ and $B = \{z / u(z) < \infty\}$

Then $G = A \cup B$ and $A \cap B = \phi$. We shall show that A and B are both open.

Let $a \in G$ and $R > 0 \Rightarrow B(a,R) \subset G$. By Harnack's inequality

$$\frac{R - |z-a|}{R + |z-a|} u_n(a) \leq u_n(z) \leq \frac{R + |z-a|}{R - |z-a|} u_n(a) \quad (1) \quad (2)$$

for $z \in B(a,R)$ and $n \geq 1$. If $a \in A$ it follows from (1) that $u_n(z) \to \infty$. Hence $B(a,R) \subset A$ which proves A is open. If $a \in B$, then (2) implies that $u(z) < \infty$ for all $z \in B(a,R)$ Hence $B(a,R) \subset B$ proving B is open.

As G is connected, either $G = A$ or $G = B$

Suppose $G = A$. Let $a \in G$ and $B(a,R) \subset G$

Let $0 < \rho < R$ Then observe that

$$\frac{R-\rho}{R+\rho} \leq \frac{R - |z-a|}{R + |z-a|} \quad \text{and} \quad \frac{R + |z-a|}{R - |z-a|} \leq \frac{R+\rho}{R-\rho}$$

for all $z \in \bar{B}(a,\rho)$.

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($z \in \bar{B}(a,\rho) \Rightarrow |z-a| \le \rho \Rightarrow R-|z-a| \ge R-\rho$ and $R+|z-a| \le R+\rho$)

Now by Harnack's inequality

$$\frac{R-\rho}{R+\rho} u_n(a) \le u_n(z) \le \frac{R+\rho}{R-\rho} u_n(a) \quad \forall z \in \bar{B}(a,\rho),\ n \ge 1$$

Left side inequality implies that $u_n(z) \to \infty$ uniformly on $\bar{B}(a,\rho)$. From this it follows that $u_n(z) \to \infty$ uniformly on compact sets (compact set is contained in a finite union of closed balls)

Now suppose $G = B$. Let $a \in G$, $B(a,R) \subset G$ and $0 < \rho < R$. Note that Harnack's inequality holds for any positive harmonic function and the constants depend only on ρ and a and not on the harmonic function. Hence for $n > m$ and $z \in \bar{B}(a,\rho)$

$$u_n(z) - u_m(z) \le \frac{R+\rho}{R-\rho}\left(u_n(a) - u_m(a)\right).$$

Since $(u_n(a))$ is Cauchy, $(u_n(z))$ is uniformly Cauchy on $\bar{B}(a,\rho)$. Hence it is uniformly Cauchy on ~~compact~~ compact sets. Therefore $(u_n(z))$ is Cauchy in $H(G)$. By part (a) $(u_n(z))$ converges in

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in Har_G But $u_n \to u$. Hence $u \in Har_G$ and this completes the proof.

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