JORDAN CANONICAL FORM

Let V be a vector space of dimension $\underline{n}$ over a field F and $T: V \to V$ be a linear transformation. $F[x]$-module structure can be defined on V using T by defining

$$f(x)v = f(T)v \quad \forall f(x) \in F[x]$$

where $f(T) = a_0 I + a_1 T + \cdots + a_n T^n$ if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

We denote this module by V_T. Since $F[x]$ is a PID and V_T is a finitely generated torsion $F[x]$-module, we have

$$V_T = \bigoplus_{i,j} \frac{F[x]}{\langle p_i^{e_{ij}} \rangle}$$

where p_i are prime elements in $F[x]$ and e_{ij} are integers. Note that p_i's are irreducible polynomials. Recall that $\prod_{i,j} p_i^{e_{ij}}$ is the characteristic polynomial of T. Suppose T has all its eigen values in F, then $p_i = x - \lambda_i$, where λ_i is an eigen value of T. Hence

$$V_T = \bigoplus_{i,j} \frac{F[x]}{\langle (x-\lambda_i)^{e_{ij}} \rangle} \qquad \text{——} (*)$$

So to study V_T it is sufficient to study $F[x]$-module of the form $\frac{F[x]}{(x-\lambda)^n}$.

This is a cyclic $F[x]$-module. Suppose v generates this. Then $\{(T-\lambda I)^{n-1}v, (T-\lambda I)^{n-2}v, \ldots (T-\lambda I)v, v\}$ is a basis of $\frac{F[x]}{(x-\lambda)^n}$ over F.

Suppose $\sum_{i=1}^{n} \alpha_i \cdot (T-\lambda I)^{n-i} v = 0$.

Multiplying through out by $(T-\lambda I)^{n-1}$ we get

$$\alpha_n (T-\lambda I)^{n-1} v = 0$$

(Note that $(T-\lambda I)^n v = 0$)

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Since $(T-\lambda I)^{n-1}v \neq 0$ we get $\alpha_n = 0$

$$\therefore \sum_{i=1}^{n-1} \alpha_i (T-\lambda I)^{n-i} v = 0$$

Now multiply throughout by $(T-\lambda I)^{n-2}$ ~~v~~. Then we get $\alpha_{n-1} = 0$. Repeating the argument we get $\alpha_i = 0$ for $i = 1, \dots, n$.

Since $\dim_F \frac{F[x]}{(x-\lambda)^n} = n$, it follows that $\{(T-\lambda I)^{n-1}v, \dots, (T-\lambda I)v, v\}$ is a basis of $\frac{F[x]}{(x-\lambda)^n}$ over F. Now the matrix of T in this basis is of the form

$$\begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & & \\ & & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}_{n\times n}$$

Such a matrix is called a Jordan block corresponding to λ. Going back to $(*)$ we get a basis of V_T (which is the union of bases of $\frac{F[x]}{\langle (x-\lambda_i)^{e_{ij}} \rangle}$) such that matrix

(4) of T with respect to this basis is of the form

$$\begin{bmatrix} J_{11} & & & & & \\ & \ddots & & & & \\ & & J_{1n_1} & & & \\ & & & J_{k1} & & \\ & & & & \ddots & \\ & & & & & J_{kn_k} \end{bmatrix}$$

where J_{ij}'s are Jordan blocks. This is called the Jordan canonical form of T.

Consider a cyclic $F[x]$-module $\frac{F[x]}{(x-\lambda)^n}$

We have shown that if v is a generator of this module, then the vectors $(T-\lambda I)^{n-1}v, (T-\lambda I)^{n-2}v, \ldots, v$ form a basis of $\frac{F[x]}{(x-\lambda)^n}$.

Defn: The set $\{(T-\lambda I)^{n-1}v, \ldots, v\}$ is called a chain of generalized eigen vectors (g.e.v) generated by v.

Remarks: (1) This is a linearly independent set of vectors.

(2) Also vectors in different chains are linearly independent (they are bases of different modules in the direct sum decomposition of V_T)

(3) The basis of V with respect to which matrix of T is in Jordan form consists of chains of g.e.vs.

This basis in (3) can be found in two ways.

(1) Finding the invariant factors of matrix A of T in some basis. This can be done by reducing the matrix $A - xI$ using elementary row (column) operations to the form

$$\begin{bmatrix} f_1(x) & & \\ & \ddots & \\ & & f_t(x) \end{bmatrix} \quad \text{with } f_1(x) \mid f_2(x) \mid \cdots \mid f_t(x)$$

and finding all prime power divisors of

(6) $f_1(x), \dots, f_t(x)$. These are the elementary divisors of T and they are of the form $(x-\lambda)^k$ where λ is an eigen value. Each elementary divisor corresponds to a Jordan block. For example $(x-\lambda)^k$ corresponds to $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & 0 & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{k \times k}$

and the chain in the basis corresponding to $(x-\lambda)^k$ is $\{(A-\lambda I)^{k-1} v, \dots, v\}$ where v is a generator of the cyclic module $\dfrac{F[x]}{(x-\lambda)^k}$

Taking the collection of all the chains corresponding to cyclic modules in $(*)$ we get the required basis.

(9) Let the characteristic polynomial of A be

$$(x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$$

Then $n = n_1 + \dots + n_k$. To get the basis of g.e.v we need to find n_i g.e.v corresponding to λ_i. Note that g.e.v belong to the $\ker(A-\lambda_i I)^k$ for some k. Find k such that $\dim \ker(A-\lambda_i I)^k = n_i$. By rank nullity theorem we have

$\operatorname{rank}(A-\lambda_i I)^k = n - n_i$. Note that ~~$k_j = \dim(A-\lambda_i I)$~~

$l_j = \dim \ker(A-\lambda_i I)^j - \dim \ker(A-\lambda_i I)^{j-1}$

is the number of linearly independent g.e.v of rank j (i.e, $(A-\lambda_i I)^j v = 0$ and $(A-\lambda_i I)^{j-1} v \neq 0$). First find l_k ~~linearly ind~~ vectors ~~of g.e.v of rank k~~ linearly independent g.e.v of rank k. Note that chains generated by these have k g.e.v of rank j for $1 \le j \le k-1$.

If $l_{k-1} = l_k$ we have already found l_{k-1} l.i g.e.v of rank $k-1$. Otherwise, find $(l_{k-1} - l_k)$ l.i g.e.v of rank $(k-1)$ which are linearly independent from l_k g.e.v of rank $(k-1)$ already found. Repeat this method for $j = k-2, k-3, \ldots, 1$.

⑥ The above methods are illustrated through the following examples:

(1) Let $A = \begin{bmatrix} 7 & 3 & 3 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix}$ over $\mathbb{Q}$

$$A - xI = \begin{bmatrix} 7-x & 3 & 3 \\ 0 & 1-x & 0 \\ -3 & -3 & 1-x \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} -3 & -3 & 1-x \\ 0 & 1-x & 0 \\ 7-x & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1(-\frac{1}{3})} \begin{bmatrix} 1 & 1 & \frac{x-1}{3} \\ 0 & 1-x & 0 \\ 7-x & 3 & 3 \end{bmatrix} \xrightarrow{R_{31}(x-7)} \begin{bmatrix} 1 & 1 & \frac{x-1}{3} \\ 0 & 1-x & 0 \\ 0 & x-4 & 3+\frac{(x-1)(x-7)}{3} \end{bmatrix}$$

$$3+\frac{(x-1)(x-7)}{3} = \frac{x^2-8x+16}{3}$$

$$\xrightarrow[C_{31}(\frac{1-x}{3})]{C_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & x-4 & \frac{x^2-8x+16}{3} \end{bmatrix} \xrightarrow{R_{23}(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & \frac{(x-4)^2}{3} \\ 0 & x-4 & \frac{(x-4)^2}{3} \end{bmatrix}$$

$$\xrightarrow{R_2(-\frac{1}{3})} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{(x-4)^2}{9} \\ 0 & x-4 & \frac{(x-4)^2}{3} \end{bmatrix} \xrightarrow{R_{32}(4-x)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{(x-4)^2}{9} \\ 0 & 0 & \frac{(x-4)^2}{3}+\frac{(x-4)^3}{9} \end{bmatrix}$$

$$= \frac{(x-4)^2}{3}\left(\frac{x-1}{3}\right)$$

$C_{32}\left(\frac{(x-4)^3}{9}\right)$, $R_3(9)$ $\longrightarrow$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x-1)(x-4)^2 \end{bmatrix}$$

$\therefore$ The invariant factors are $1, 1, (x-1)(x-4)^2$

Elementary divisors are $(x-1)$ and $(x-4)^2$

Rational canonical form of $A = C\left((x-1)(x-4)^2\right)$, the companion matrix of the polynomial $(x-1)(x-4)^2$

which is $x^3 - 9x^2 + 24x - 16$

$\therefore$ Rational canonical form of $A = \begin{bmatrix} 0 & 0 & +16 \\ 1 & 0 & -24 \\ 0 & 1 & 9 \end{bmatrix}$

Jordan canonical form of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

Method 2: First find the eigen values of A

They are given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & 3 & 3 \\ 0 & 1-\lambda & 0 \\ -3 & -3 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 8\lambda + 16) = (1-\lambda)(\lambda-4)^2$$

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Hence eigen values are 1 and 4 (mult 2)

We have to find one gev of rank 1 Corresponding to 1

and 2 l i g.e.vs corresponding to 4. Note that

gev of rank 1 is simply an eigen vector.

Eigen vector corresponding to 1:

$$(A - I)X = 0 \implies \begin{bmatrix} 6 & 3 & 3 \\ 0 & 0 & 0 \\ -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Clearly $X = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ satisfies the above equation.

Now g.e.vs corresponding to 4

$$(A - 4I) = \begin{bmatrix} 3 & 3 & 3 \\ 0 & -3 & 0 \\ -3 & -3 & -3 \end{bmatrix}$$ has rank 2 (two rows are l.i)

and $$(A-4I)^2 = \begin{bmatrix} 0 & -9 & 0 \\ 0 & 9 & 0 \\ 0 & 9 & 0 \end{bmatrix}$$ has rank $1 = 3-2$
$= 3 -$ mult of 4.

~~This~~

Find $Y \ni (A-4I)^2 Y = 0$ but $(A-4I)Y \neq 0$.

Clearly $Y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies these conditions.

$\therefore Y_2$ is a g.e.v of rank 2.

Let $Y_1 = (A-4I)Y_2 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$.

Now $\{X, Y_1, Y_2\}$ is a basis consisting of g.e.vs.

Let $M = [X \; Y_1 \; Y_2]$, the matrix whose columns are X, Y_1, Y_2.

Now $AM = \begin{bmatrix} 1 & 12 & 7 \\ -1 & 0 & 0 \\ -1 & -12 & -3 \end{bmatrix}$ and $M \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{J \text{ say}} = \begin{bmatrix} 1 & 12 & 7 \\ -1 & 0 & 0 \\ -1 & -12 & -3 \end{bmatrix}$

Hence $AM = MJ$ or $M^{-1}AM = J$, the Jordan form of A.

(2) Let $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

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(13) Then

$$A - xI = \begin{bmatrix} 2-x & 1 & 0 & -1 \\ 0 & 2-x & 1 & 1 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{bmatrix} \xrightarrow{C_{12}} \begin{bmatrix} 1 & 2-x & 0 & -1 \\ 2-x & 0 & 1 & 1 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{bmatrix}$$

$$\xrightarrow{R_{21}(x-2)} \begin{bmatrix} 1 & 2-x & 0 & -1 \\ 0 & -(x-2)^2 & 1 & 3-x \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{bmatrix} \xrightarrow[C_{41}(1)]{C_{21}(x-2)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(x-2)^2 & 1 & 3-x \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{bmatrix}$$

$$\xrightarrow{C_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(x-2)^2 & 3-x \\ 0 & 2-x & 0 & 0 \\ 0 & 0 & 0 & 2-x \end{bmatrix} \xrightarrow[C_{42}(x-3)]{C_{32}((x-2)^2)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2-x & -(x-2)^3 & (x-3)(x-2) \\ 0 & 0 & 0 & 2-x \end{bmatrix}$$

$$\xrightarrow{C_{32}(x-2)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(x-2)^3 & -(x-3)(x-2) \\ 0 & 0 & 0 & 2-x \end{bmatrix} \xrightarrow{C_{34}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(x-3)(x-2) & -(x-2)^3 \\ 0 & 0 & 2-x & 0 \end{bmatrix}$$

$$\xrightarrow{R_{34}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & -(x-3)(x-2) & -(x-2)^3 \end{bmatrix} \xrightarrow{R_{43}(3-x)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & -(x-2)^3 \end{bmatrix}$$

$$\xrightarrow{R_4(-1)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & (x-2)^3 \end{bmatrix}$$

Hence the invariant factors of A are

$1, 1, x-2, (x-2)^3$

$\therefore$ Rational canonical form of A

$$= \begin{bmatrix} C(x-2) & \\ & C(x-2)^3 \end{bmatrix}$$

$(x-2)^3$
$= x^3 - 6x^2 + 12x - 8$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{bmatrix} 8 \\ -12 \\ 6 \end{bmatrix} \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

The Jordan canonical form $= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$= J$ (say)

<u>Method 2:</u> Since A is an upper triangular matrix its eigen values are diagonal entries. Hence 2 is the only eigen value with multiplicity 4. So find k such that $(A-2I)^k$ has rank $= 4-4=0$.

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$$A-2I = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has rank 2 ('2nd and 3rd columns are l.i and 4th column is a linear combination of 2nd & 3rd columns).

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has rank 1

$(A-2I)^3 = 0$ has rank 0

Find x such that $(A-2I)^3 x = 0$ but $(A-2I)^2 x \neq 0$

clearly $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ satisfies these conditions.

$$\{(A-2I)^2 x_3, (A-2I)x_3, x_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\dim\ker(A-2I) = 2 \Rightarrow$ we need 2 l.i eigen vectors

$(A-2I)^2 x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is one eigen vector.

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So we find Y such that $(A-2I)Y=0$ and Y is l.i from $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

Note that $Y = \begin{bmatrix}0\\1\\-1\\+1\end{bmatrix}$ satisfies this condition.

Hence the required basis is $\{Y, (A-2I)^3X_3, (A-2I)X_3, X_3\}$.

Let $M = \begin{bmatrix} Y & (A-2I)^3X_3 & (A-2I)X_3 & X_3 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It can be checked that $AM = MJ$.

Assignment: Find the rational canonical form and the Jordan canonical form and the basis of g.e vs for the following matrices:

(1) $\begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} 3 & 1 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

The above assignment is against the 3rd internal

Submit this with in a week so that I get time

21.5.20

to check.

You can submit through Email: knn_k@yahoo.co.in

Assignment for M214 (complex analysis)

I have sent notes for Riemann Zeta function

Runge's theorem, Analytic continuation and Harmonic

functions.

Assignment identify all the theorems/results used

in the proofs and state them.

Assignment due date. 15.6.20.

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